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MATHEMATICAL ASPECTS OF ELECTROMAGNETIC THEORY III

By

Bernard Friedman

Technical Report No. 4

Prepared under Contract Nonr 222(60)

(NR 041-221)

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MATHEMATICAL ASPECTS OF ELECTROMAGNETIC THEORY 111

I. Bessel Functions

In this section we shall investigate the Bessel functions by studying the solutions of the two dimensional wave equation written in polar form. No prior knowledge of Bessel functions will be assumed; their properties and representations will arise as a natural consequence of the by now familiar methods of solution.

The wave equation in rectangular coordinates,

$$\varphi_{xx} + \varphi_{yy} + k^2\varphi = -\delta(x - x_0)\delta(y - y_0),$$

is simply a generalization of equation (33) Technical Report No. 2 and its solution is easily seen to be equation (34) written in a slightly more general form:

$$\varphi(x,y) = -\frac{1}{4\pi i} \int_C \frac{e^{ip(x-x_0) + i\sqrt{k^2 - p^2}|y-y_0|}}{\sqrt{k^2 - p^2}} dp ,$$

where the contour C is identical with the contour used in (34).

However, the wave equation may be solved in any coordinate system which permits the separation of variables. Thus we can introduce plane polar coordinates and solve

$$\varphi_{rr} + \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi_{\theta\theta} + k^2\varphi = -\frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r} .$$

In order to accomplish this we will try a solution of the form $\varphi(r, \theta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} u_n(r) e^{in\theta}$. This expression for $\varphi(r, \theta)$ implies that, for $m = 0, 1, 2$,

$$\frac{d^m}{dr^m} u_n(r) = \int_0^{2\pi} \frac{\partial^m}{\partial r^m} \varphi(r, \theta) e^{-in\theta} d\theta;$$

also, integration by parts shows that

$$\int_0^{2\pi} \varphi_{\theta\theta} e^{-in\theta} d\theta = (\varphi_{\theta} + in\varphi) e^{-in\theta} \Big|_0^{2\pi} - \int_0^{2\pi} n^2 \varphi e^{-in\theta} d\theta = -n^2 u_n(r),$$

since, from physical considerations, we can assume $\varphi(r, 0) = \varphi(r, 2\pi)$ and $\varphi_{\theta}(r, 0) = \varphi_{\theta}(r, 2\pi)$. Thus, when the wave equation is multiplied by $e^{-in\theta}$ and both sides are integrated over $0 \leq \theta \leq 2\pi$, we have

$$(1) \quad u_n'' + \frac{1}{r} u_n' + (k^2 - \frac{n^2}{r^2}) u_n = - \frac{\int (r-r_0)}{r} e^{-in\theta_0}, \quad 0 \leq r \leq \infty.$$

There is evidently only one boundary condition for equation (1), namely, $u_n(r)$ must satisfy the radiation condition and be outgoing at infinity. A boundary condition at $r = 0$ is necessary in order to define a unique solution. Mathematically, the change to polar coordinates causes the origin $r = 0$ to become a singular point of the differential equation (1). As a consequence, not all the solutions of (1) are regular at $r = 0$. But physically the origin is a point in space

without special distinction, and the field produced by the line source along (r_0, θ_0) should be defined here. Therefore our second condition is that the solution of (1) which we select must have no singularity at the origin.

The usual Green's function technique will be applied to equation (1). The homogeneous equation is

$$w_n''(r) + \frac{1}{r}w_n'(r) + (k^2 - \frac{n^2}{r^2})w_n(r) = 0.$$

By setting $kr = \rho$, we obtain Bessel's equation

$$(2) \quad w_n''(\rho) + \frac{1}{\rho}w_n'(\rho) + (1 - \frac{n^2}{\rho^2})w_n(\rho) = 0,$$

where the primes now denote differentiation with respect to ρ . Attempting a power series solution of the form

$$w = \rho^\alpha (1 + a_1 \rho + a_2 \rho^2 + \dots)$$

we find after substitution into (2) that the indicial equation is $\alpha^2 - n^2 = 0$, i.e., $\alpha = \pm n$. Thus the two fundamental solutions of equation (2) behave as ρ^n and ρ^{-n} . The solution that behaves as $\rho^{-|n|}$ will be discarded because it is not regular at $\rho = 0$; the other solution will be labeled $J_n(\rho)$, $n \geq 0$, and called the Bessel function of order n . Consequently, for $r < r_0$, $u_n(kr) = w_n(kr) = \text{const } J_n(kr) \sim (kr)^{|n|}$ (\sim denotes "behaves as").

Notice that for $n = 0$ it appears the homogeneous equation

has only one solution; however, in this case it can be shown that a second solution exists which behaves like $\log \rho$. In any case, the equation (2) always has two linearly independent solutions.

In order to determine which solution of equation (2) is outgoing at infinity, we make the substitution $w_n = \rho^{-1/2} v_n$; this will remove the first derivative term in (2). (Such a removal can always be accomplished in a second order differential equation by means of the general substitution $w = fv$, where f is a suitably chosen function.) The homogeneous equation becomes

$$v_n''(\rho) + \left(1 - \frac{n^2 - 1/4}{\rho^2}\right)v_n(\rho) = 0$$

Since we are interested in $v_n(\rho)$ for large values of ρ , we neglect the term $\frac{n^2 - 1/4}{\rho^2}$, and thus obtain $v_n \sim e^{\pm i\rho}$ and $w_n \sim \rho^{-1/2} e^{\pm i\rho}$ for ρ large. The outgoing and incoming solutions of equation (2) are called the Hankel functions and defined as follows: $H_n^{(1)}(\rho) \sim \rho^{-1/2} e^{i\rho}$, $H_n^{(2)}(\rho) \sim \rho^{-1/2} e^{-i\rho}$ where $n \geq 0$. Recall that the factor $\rho^{-1/2} = (kr)^{-1/2}$ has also appeared in earlier investigations of the behavior of cylindrical waves at large distances from their source and that this factor is compatible with the conservation of field

energy (e.g., see section VIII in Technical Report No. 2).

Proceeding with the Green's function technique, we put

$$\begin{aligned} u_n(r) &= \text{const } J_n(r), & r < r_0 \\ &= \text{const } H_n^{(1)}(kr), & r > r_0. \end{aligned}$$

When we cross multiply in order to assure the continuity of $u_n(r)$, we find

$$\begin{aligned} u_n(r) &= c_n H_n^{(1)}(kr_0) J_n(kr), & r < r_0 \\ &= c_n J_n(kr_0) H_n^{(1)}(kr), & r > r_0. \end{aligned}$$

The constants c_n are to be determined by the jump condition,

$$u'_n(r) \Big|_{r_0+} - u'_n(r) \Big|_{r_0-} = \frac{-ie^{-i\theta_0}}{r_0}$$

i.e., the c_n should be such that

$$(3) \quad c_n [J_n(kr_0) \frac{d}{dr} H_n^{(1)}(kr) - H_n^{(1)}(kr_0) \frac{d}{dr} J_n(kr)]_{r=r_0} = \frac{-ie^{-i\theta_0}}{r_0}.$$

If we let the prime denote differentiation with respect to

kr , equation (3) can be rewritten

$$c_n k [J_n H_n^{(1)'} - H_n^{(1)} J_n']_{r=r_0} = \frac{-ie^{-i\theta_0}}{r_0}.$$

The bracketed expression is essentially the Wronskian of the two chosen solutions of the homogeneous equation. It should not be surprising that $[r J_n H_n^{(1)'} - r H_n^{(1)} J_n']_{r=r_0} = \text{const } e^{-i\theta_0}$ is independent of r_0 , for if equation (2) is written in self-adjoint form,

$$(\rho w_n')' + (\rho - \frac{n^2}{\rho})w_n = 0,$$

we find

$$(\rho J_n')' + (\rho - \frac{n^2}{\rho})J_n = 0$$

and

$$(\rho H_n^{(1)'})' + (\rho - \frac{n^2}{\rho})H_n^{(1)} = 0$$

$$\Rightarrow H_n^{(1)}(\rho J_n')' - J_n(\rho H_n^{(1)'})' = 0$$

$$\Rightarrow \frac{d}{d\rho}[\rho J_n' H_n^{(1)} - \rho H_n^{(1)'} J_n] = 0$$

$$\Rightarrow \rho [J_n' H_n^{(1)} - H_n^{(1)'} J_n] = \text{const.}$$

Because the Wronskian is a constant, the next step would ordinarily be to evaluate J_n , $H_n^{(1)}$, and their derivatives at some convenient value of r_0 , substitute back into equation (3), and thus find the c_n . Usually r_0 is taken large since the asymptotic behaviors of the Bessel and Hankel functions are well known and easy to work with. But because we assume no knowledge of these functions, their behavior, asymptotic or otherwise, must first be determined as part of the solution of equation (1).

To begin with, recurrence relations can be obtained for the solutions $w_n(kr)$ of the homogeneous equation. We know $w_n(kr)e^{in\theta}$ satisfies $\phi_{xx} + \phi_{yy} + k^2\phi = 0$; we notice also that ϕ_x , ϕ_y , and any linear combinations thereof are solutions. Therefore, the combinations

$$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})w_n(kr)e^{in\theta} \text{ and } (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})w_n(kr)e^{in\theta}$$

satisfy $\phi_{xx} + \phi_{yy} + k^2\phi = 0$. If $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are expressed in polar coordinates we find that

$$\left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial}{\partial \theta}\right) w_n(\rho) e^{i(n+1)\theta} \text{ and } \left(\frac{\partial}{\partial \rho} - \frac{1}{\rho} \frac{\partial}{\partial \theta}\right) w_n(\rho) e^{i(n-1)\theta}$$

must each be solutions of the wave equation. Consider the

first of these, which reduces to $(w'_n - \frac{n}{\rho} w_n) e^{i(n+1)\theta}$:

by referring back to the derivation of equation (1), it is

not difficult to see that $(w'_n - \frac{n}{\rho} w_n)$ solves equation (2)

when n has been replaced by $n+1$ in that equation. We con-

clude that for some constant α ,

$$w'_n - \frac{n}{\rho} w_n = \alpha w_{n+1}.$$

Similarly, using the second of the linear combinations

above, we conclude that

$$w'_n + \frac{n}{\rho} w_n = \beta w_{n-1}.$$

The quantities α and β are yet to be specified; to this

end, note that $w'_{n+1} + \frac{n+1}{\rho} w_{n+1} = \beta w_n$, from which follows

$$w''_n + \frac{1}{\rho} w'_n - \frac{n^2}{\rho^2} w_n = \alpha \beta w_n.$$

Because w_n satisfies the homogeneous equation (2), we have

$\alpha \beta w_n = -w_n$ and $\alpha \beta = -1$. We shall choose $\alpha = -1$, $\beta = 1$.

The recurrence relations are therefore

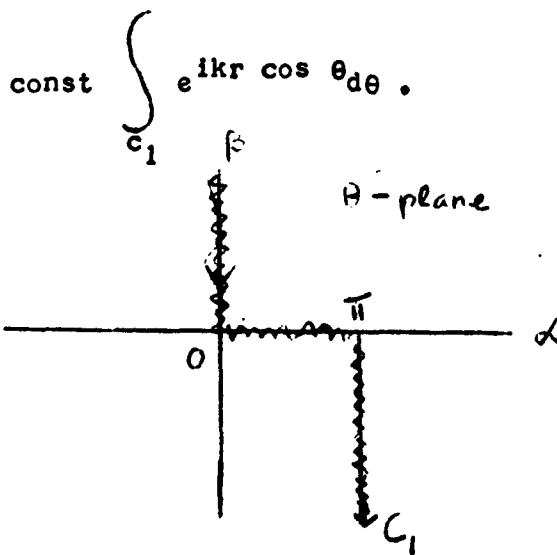
$$(4) \quad w'_n - \frac{n}{\rho} w_n = -w_{n+1}, \quad w'_n + \frac{n}{\rho} w_n = w_{n-1}.$$

Recall that J_n , $H_n^{(1)}$, and $H_n^{(2)}$ are solutions of the homogeneous equation and so are qualified for the above relations; thus, for example, if $H_0^{(1)}$ can be determined, the above formulae may be used to write $H_n^{(1)}$ for any n . This fact will enable us to arrive at an integral representation for $H_n^{(1)}$ as soon as we've found one for $H_0^{(1)}$; from this integral representation will follow the asymptotic behaviors of $H_n^{(1)}$, $H_n^{(2)}$, and J_n , which in turn will enable us to compute the c_n .

Since $J_n(kr) \sim (kr)^{|n|}$, $J_n(0) = 0$ for $n \neq 0$. We shall specify that $\underline{J_0(0) = 1}$. It is known from Section VIII that for $x_0 = y_0 = 0$, the solution of

$$\phi_{xx} + \phi_{yy} + k^2 \phi = -\delta(x-x_0) \delta(y-y_0)$$

can be written

$$\phi = \text{const} \int_{C_1} e^{ikr \cos \theta} d\theta.$$


The diagram illustrates the H-plane with a coordinate system. The horizontal axis is labeled x and the vertical axis is labeled y . The origin is marked with O . A contour C_1 is depicted in the first quadrant, consisting of a vertical segment on the y -axis, a horizontal segment, a vertical segment, and a horizontal segment along the x -axis. The label "H-plane" is placed in the upper right area of the diagram.

On the other hand, if $r_0 = 0$ we have seen that

$$\varphi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} a_n J_n(kr_0) H_n^{(1)}(kr) e^{in\theta} \quad n > r.$$

Hence, because $J_n(0) = 0$ for $n \neq 0$ and $J_0(0) = 1$, we must have

$$H_0^{(1)}(kr) = K \int_{C_1} e^{ikr \cos \theta} d\theta,$$

where K is some constant. Now, $H_n^{(1)}(kr) \sim (kr)^{-1/2} e^{ikr}$ and $H_n^{(2)}(kr) \sim (kr)^{-1/2} e^{-ikr}$, which implies that

$$H_0^{(2)}(kr) = \overline{H_0^{(1)}(kr)} = K \int_{C_1} e^{-ikr \cos \theta} d\theta.$$

We wish to compute the complex conjugate of the integral, a task which is complicated by the fact that C_1 lies in the complex plane. $H_0^{(1)}$ may be reduced to the sum of the three following integrals over real paths by the substitution

$$\theta = \alpha + i\beta :$$

$$H_0^{(1)} = -iK \int_{-\infty}^0 e^{ikr \cosh \beta} d\beta + K \int_0^{\pi} e^{ikr \cos \alpha} d\alpha + iK \int_0^{\infty} e^{-ikr \cosh \beta} d\beta,$$

therefore

$$H_0^{(2)} = iK \int_{-\infty}^0 e^{-ikr \cosh \beta} d\beta + K \int_0^{\pi} e^{-ikr \cos \alpha} d\alpha - iK \int_0^{\infty} e^{ikr \cosh \beta} d\beta$$

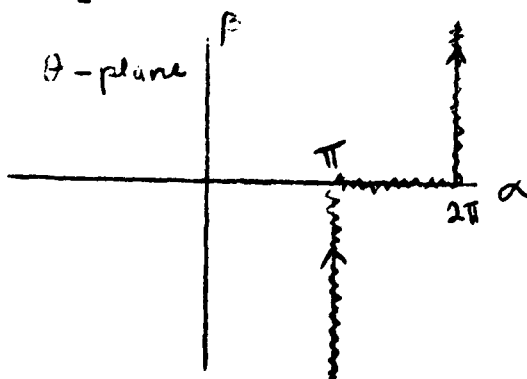
Letting $\beta' = -\beta$, $\alpha' + \pi = \alpha$, $H_0^{(2)}$ can be written as

$$H_0^{(2)} = iK \int_{-\infty}^0 e^{ikr \cosh \beta'} d\beta' + K \int_{\pi}^{2\pi} e^{ikr \cos \alpha'} d\alpha' + \\ + iK \int_0^{\infty} e^{ikr \cosh \beta'} d\beta' .$$

Thus

$$H_0^{(2)}(kr) = K \int_{C_2} e^{ikr \cos \theta} d\theta ,$$

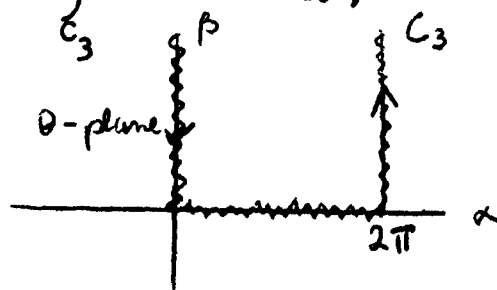
where the contour C_2 is as follows:



Furthermore,

$$H_0^{(1)}(kr) + H_0^{(2)}(kr) = K \left(\int_{C_1} + \int_{C_2} \right) e^{ikr \cos \theta} d\theta =$$

$$= K \int_{C_3} e^{ikr \cos \theta} d\theta ,$$



But because the integrand is periodic with respect to α , the integrals taken over the vertical portions of C_3 cancel one another. Therefore,

$$H_0^{(1)}(kr) + H_0^{(2)}(kr) = K \int_0^{2\pi} e^{ikr \cos \alpha} d\alpha.$$

The above sum is a solution of equation (2); it is real since $H_0^{(1)}$ and $H_0^{(2)}$ are complex conjugates, and it is regular at the origin. We conclude that $H_0^{(1)} + H_0^{(2)}$ is equal to some multiple of J_0 ; we define

$$H_0^{(1)}(kr) + H_0^{(2)}(kr) = 2J_0(kr).$$

From the fact that $J_0(0) = 1$, we find that $K = \pi^{-1}$. We now have the desired integral representations:

$$(5) \quad H_0^{(1)}(kr) = \frac{1}{\pi} \int_{C_1} e^{ikr \cos \theta} d\theta, \quad H_0^{(2)}(kr) = \frac{1}{\pi} \int_{C_2} e^{ikr \cos \theta} d\theta,$$

and

$$J_0(kr) = \frac{1}{\pi} \int_0^{2\pi} e^{ikr \cos \alpha} d\alpha.$$

Note: Since $H_n^{(1)}$, $H_n^{(2)}$, and J_n all satisfy the recursion relations (4), we see that $H_0^{(1)} + H_0^{(2)} = 2J_0 \Rightarrow H_n^{(1)} + H_n^{(2)} = 2J_n$. The difference of the two n^{th} -order Hankel functions is usually denoted by $2N_n = H_n^{(1)} - H_n^{(2)}$ and is known as the Neumann function of order n .

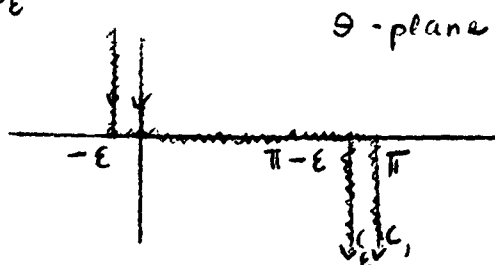
It shall now be shown that the outgoing Hankel function of order n is given by

$$(6) \quad H_n^{(1)}(\rho) = \frac{(-1)^n}{\pi} \int_{C_1} e^{i\rho \cos \theta} e^{in\theta} d\theta.$$

Equation (6) will follow directly from the fact that the integral (5) satisfies the recursion formula (4) for H_n , namely

$$-H_{n+1}^{(1)} = H_n^{(1)'} - \frac{n}{\rho} H_n^{(1)}.$$

In order to prove this fact, we shift the path of integration from C_1 to C_ξ



In Section VIII it was shown that for $0 \leq \xi \leq \pi$,

$$\int_{C_1} e^{ikr \cos \theta} d\theta = \int_{C_\xi} e^{ikr \cos \theta} d\theta,$$

and a minor rearrangement of that proof also gives

$$\int_{C_1} e^{ikr \cos \theta} e^{in\theta} d\theta = \int_{C_\xi} e^{ikr \cos \theta} e^{in\theta} d\theta.$$

Let $\frac{(-1)^n}{\pi} \int_{C_\xi} e^{i\rho \cos \theta} e^{in\theta} d\theta = I_{\xi,n}(\rho) \quad (\rho = kr);$ then

since \int_{C_ξ} is uniformly convergent for $0 < \xi < \pi$,

$$\begin{aligned}
 I'_{\xi, n} - \frac{n}{\rho} I_{\xi, n} &= \frac{(-1)^n}{\pi} \int_{C_\xi} e^{i\rho \cos \theta} e^{in\theta} \left[1 \cos \theta - \frac{n}{\rho}\right] d\theta = \\
 &= \frac{(-1)^n}{\pi} \int_C e^{i\rho \cos \theta} e^{in\theta} [1 \cos \theta - \sin \theta] d\theta + \\
 &+ \frac{(-1)^n}{\pi} \left. \frac{e^{i\rho \cos \theta} e^{in\theta}}{-i\rho} \right|_{C_\xi} \text{ where we have integrated by parts.}
 \end{aligned}$$

Since the first term here equals $-I_{\xi, n+1}$, it need only be shown that the integrated part is zero. But

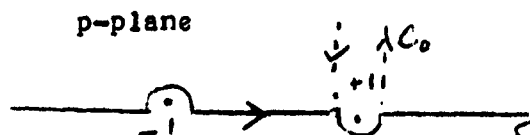
$$\begin{aligned}
 \left. e^{i\rho \cos \theta} e^{in\theta} \right|_{C_\xi} &= e^{i\rho \cos(-\xi + i\beta)} e^{in(-\xi + i\beta)} \Big|_{\infty}^0 + \\
 &+ e^{i\rho \cos \xi} e^{in\xi} \Big|_{-\xi}^{\pi-\xi} + \dots + e^{i\rho \cos(\pi-\xi + i\beta)} e^{in(\pi-\xi + i\beta)} \Big|_0^{-\infty} =
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\beta \rightarrow \infty} [\exp(-i\rho \cos \xi \cosh \beta - \rho \sin \xi \sinh \beta + in(\pi-\xi) + n\beta) + \dots \\
 &\dots + \exp(i\rho \cos(\xi - i\beta) - in\xi - n\beta)].
 \end{aligned}$$

This limit is equal to zero so long as $\pi > \xi > 0$, for in the first term $-\rho \sin \xi \sinh \beta + n\beta \sim -\rho \sin \xi e^\beta + n\beta \rightarrow -\infty$ as $\beta \rightarrow \infty$, and in the second term $i\rho \cos(\xi - i\beta) - n\beta = i\rho \cos \xi \cosh \beta - \rho \sin \xi \sinh \beta - n\beta \sim -\rho \sin \xi e^\beta - n\beta \rightarrow -\infty$ as $\beta \rightarrow \infty$. Equation (6) is thereby proved.

Equation (6) enables us, after some familiar preliminary manipulations, to work out an asymptotic expression for $H_n^{(1)}(\rho)$. If we let $p = \cos \theta$, (6) becomes

$$H_n^{(1)}(\rho) = \frac{(-i)^n}{\pi} \int_C e^{i\rho p} e^{in\theta} \frac{dp}{\sqrt{1-p^2}},$$



As before, in order to obtain an integrand which decreases as p increases, the contour C is shifted to C_0 and p put equal to $1+is$ (notice that $e^{in\theta}$ is analytic except at $p = \pm 1$). It follows that

$$H_n^{(1)}(\rho) = \frac{2}{\pi} e^{i(\rho - \pi/4 - n\pi/2)} \int_0^\infty \frac{e^{-\rho s} e^{in\theta}}{\sqrt{s} \sqrt{2+is}} ds.$$

Since $e^{in\theta}$ has no derivative at $s = 0$ ($\frac{d}{ds} e^{in\theta} = in e^{in\theta} [- \frac{1}{\sqrt{s^2 - 2is}}]$) we may not legitimately use the theorem of Section VIII. However, at the end of that section an alternate method was used to arrive at an asymptotic approximation of E_z , and this method will also find application here.

The value of $e^{in\theta} (2+is)^{-1/2}$ at $s = 0$ is $1 \cdot (2)^{-1/2}$; we can write

$$\frac{e^{in\theta}}{\sqrt{2+is}} = \frac{1}{\sqrt{2}} + \left[\frac{\sqrt{2} e^{in\theta} - \sqrt{2+is}}{\sqrt{2} \sqrt{2+is}} \right].$$

Substituting this into the expression for $H_n^{(1)}$ and integrating the leading term, we have

$$H_n^{(1)}(\rho) = \sqrt{\frac{2}{\pi}} e^{i(\rho - \pi/4 - n\pi/2)} + E,$$

where

$$E = \sqrt{\frac{2}{\pi}} e^{i(\rho - \pi/4 - n\pi/2)} \int_0^\infty \frac{e^{-\rho s}}{\sqrt{s}} \left[\frac{\sqrt{2} e^{in\theta} - \sqrt{2+is}}{\sqrt{2}\sqrt{2+is}} \right] ds.$$

We may write

$$\frac{\sqrt{2} e^{in\theta} - \sqrt{2+is}}{\sqrt{2}\sqrt{2+is}} = \frac{e^{in\theta} - 1}{\sqrt{2+is}} + \frac{\sqrt{2} - \sqrt{2+is}}{\sqrt{2}\sqrt{2+is}}.$$

The second term on the right, which also appeared in Section VII

is absolutely bounded by $\frac{s}{4\sqrt{2}}$. A bound for the first term

can also be obtained. Recalling that $\cos\theta = 1 + is$, it is

easy to see that $(e^{i\theta} - 1) = \sqrt{2is} e^{i\theta/2}$; and because

$$(e^{in\theta} - 1) = (e^{i\theta} - 1)(e^{i(n-1)\theta} + e^{i(n-2)\theta} + \dots + e^{i\theta} + 1),$$

it will follow that $|e^{in\theta} - 1| \leq \sqrt{2} n \sqrt{s}$. Hence,

$$|E| \leq \frac{2}{\pi} \int_0^\infty \frac{e^{-\rho s}}{\sqrt{s}} \left(\frac{\sqrt{2} n \sqrt{s}}{\sqrt{2}} \right) ds + \frac{2}{\pi} \int_0^\infty \frac{e^{-\rho s}}{\sqrt{s}} \left(\frac{s}{4\sqrt{2}} \right) ds,$$

$$|E| \leq \text{const } \frac{n}{\rho} + \text{const } \frac{1}{\rho^{3/2}}.$$

This error estimate is satisfactory whenever $n \ll \rho$ (the

behavior of $H_n^{(1)}$ for large n will be investigated later).

Thus, since $H_n^{(2)}(\rho) = H_n^{(1)}(\rho)$ and $2J_n(\rho) = H_n^{(1)}(\rho) + H_n^{(2)}(\rho)$,

we have determined the asymptotic behavior of the Bessel

and Hankel functions:

$$(7) \begin{cases} H_n^{(1)}(\rho) \sim \sqrt{\frac{2}{\pi\rho}} e^{i(\rho - \pi/4 - n\pi/2)} \\ H_n^{(2)}(\rho) \sim \sqrt{\frac{2}{\pi\rho}} e^{-i(\rho - \pi/4 - n\pi/2)} \\ J_n(\rho) \sim \sqrt{\frac{2}{\pi\rho}} \cos(\rho - \pi/4 - n\pi/2) \end{cases} \quad (\rho \gg n > 0)$$

We are now finally in a position to compute the c_n of equation (3) and thus complete our solution. In general, asymptotic forms may not be differentiated to yield the asymptotic forms of the derivatives. However, in this case, since the paths C_1 and C_2 which appear in the integral expressions of $H_n^{(1)}$ and $H_n^{(2)}$ may be slightly altered in order to assure the absolute convergence of the integrals, equation (6) can be differentiated with respect to ρ . The estimation of $H_n^{(1)'}(\rho)$ may be then carried out, and by referring to the derivation of the asymptotic form of $H_n^{(1)}(\rho)$ it is almost immediate that

$$H_n^{(1)'}(\rho) \sim iH_n^{(1)}(\rho) + O(\rho^{-1}).$$

Therefore,

$$H_n^{(1)'}(\rho) \sim i\sqrt{\frac{2}{\pi}} e^{i(\rho - \pi/4 - n\pi/2)} = iH_n^{(1)}(\rho)$$

$$J_n'(\rho) \sim -\sqrt{\frac{2}{\pi}} \sin(\rho - \pi/4 - n\pi/2),$$

where the terms containing ρ^{-1} have been dropped. When these asymptotic forms are substituted into equation (3),

we find $c_n \left[\frac{2i}{\pi} \right] = -e^{-in\theta_0}$.

The solution $\varphi = \sum_{-\infty}^{\infty} u_n(r) e^{in\theta}$ of the wave equation is

$$\begin{aligned} \varphi(r, \theta) &= \frac{1}{4} \sum_{-\infty}^{\infty} H_{|n|}^{(1)}(kr_0) J_{|n|}(kr) e^{in(\theta-\theta_0)}, \quad r < r_0 \\ &= \frac{1}{4} \sum_{-\infty}^{\infty} J_{|n|}(kr_0) H_{|n|}^{(1)}(kr) e^{in(\theta-\theta_0)}, \quad r > r_0 \end{aligned}$$

It may be more simply written by defining

$$r_> \equiv \max(r, r_0), \quad r_< \equiv \min(r, r_0):$$

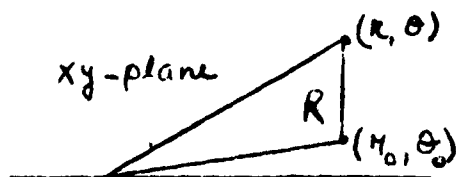
we get

$$(8) \quad \varphi(r, \theta) = \frac{1}{4} \sum_{-\infty}^{\infty} H_{|n|}^{(1)}(kr_>) J_{|n|}(kr_<) e^{in(\theta-\theta_0)}.$$

* * *

When the integral expression for $H_0^{(1)}(kr)$ was being computed we saw that for $r_0 = 0$, $\varphi(r, \theta) = \frac{c_0}{2\pi} H_0^{(1)}(kr)$; c_0 is known from $c_n = \frac{\pi i}{2} e^{-in\theta_0}$. Therefore, if (r_0, θ_0) on the xy-plane is taken as the origin we can write

$$(9) \quad H_0^{(1)}(kr) = \sum_{-\infty}^{\infty} H_{|n|}^{(1)}(kr_>) J_{|n|}(kr_<) e^{in(\theta-\theta_0)},$$



Equation (9) leads to the evaluation of a definite integral.

Since $R = \sqrt{r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0)}$, it follows that

$$\begin{aligned} (10) \quad \frac{1}{2\pi} \int_0^{2\pi} H_0^{(1)}(k \sqrt{r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0)}) e^{-in\theta} d\theta &= \\ &= H_{|n|}^{(1)}(kr_0) J_{|n|}(kr) e^{-in\theta_0}. \end{aligned}$$

* * *

We might ask what happens when the line source is at infinity, i.e., when $r_0 \rightarrow \infty$. As r_0 becomes large,

$$\begin{aligned} R &= r_0 \left[1 + \frac{r^2}{r_0^2} - \frac{2r}{r_0} \cos(\theta - \theta_0) \right]^{1/2} = \\ &= r_0 \left[1 - \frac{r}{r_0} \cos(\theta - \theta_0) \right] + O(r_0^{-1}), \end{aligned}$$

and referring to the asymptotic form of $H_0^{(1)}(kR)$, we have

$$H_0^{(1)}(kR) = \left(\frac{2}{\pi kR}\right)^{1/2} e^{i(kR - \pi/4)} + O(R^{-1})$$

$$(11) \quad = \left(\frac{2}{\pi k r_0}\right)^{1/2} e^{i(kr_0 - \pi/4)} e^{-ikr \cos(\theta - \theta_0)} + O(r_0^{-1})$$

Since the function $e^{-ikr \cos(\theta - \theta_0)}$ represents an incoming plane wave from the direction θ_0 , we see that the line source at very large distance from the origin is asymptotically a plane wave of very small amplitude $(r_0^{-1/2})$.

Consider the right-hand side of equation (9). Since r is fixed and $r_0 \rightarrow \infty$, we have $r_> = r_0$ and $r_< = r$. With the aid of estimates of $H_n^{(1)}$ and J_n for large values of n that we shall obtain later, it is easy to show that the series in (9) is absolutely and uniformly convergent. In fact, we can show that

$$(12) \quad \sum_{n=-\infty}^{\infty} = \sum_{n=-N}^N H_{|n|}^{(1)}(kr_0) J_{|n|}(kr) e^{in(\theta-\theta_0)} + O(r_0^{-N-1})$$

If r_0 is large compared to N , equation (7) gives

$$(13) \quad H_{|n|}^{(1)}(kr_0) = \left(\frac{2}{\pi kr_0}\right)^{-1/2} e^{i(kr_0 - \pi/4 - |n|\pi/2)} + O(r_0^{-1})$$

Substituting (11) and (13) in (9), we obtain

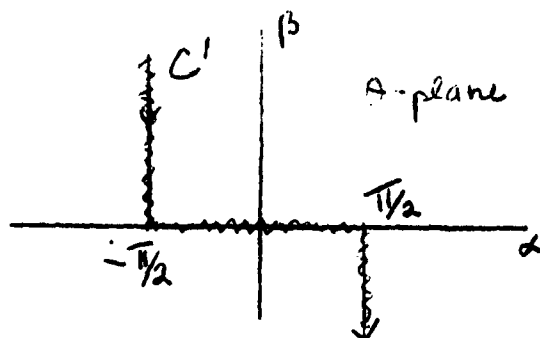
$$(14) \quad e^{-ikr \cos(\theta-\theta_0)} = \sum_{n=-\infty}^{\infty} (-1)^n J_{|n|}(kr) e^{in(\theta-\theta_0)},$$

a representation of a plane wave as a sum of incoming and outgoing cylindrical waves.

* * *

It still remains to discuss the asymptotic behavior of $H_n^{(1)}(\rho) = \frac{(-1)^n}{\pi} \int_{C_1} e^{i\rho \cos \theta} e^{in\theta} d\theta$ for the case of large n .

(the path C_1 is shown on page 12) . Instead of C_1 , however, we shall use the contour C' :

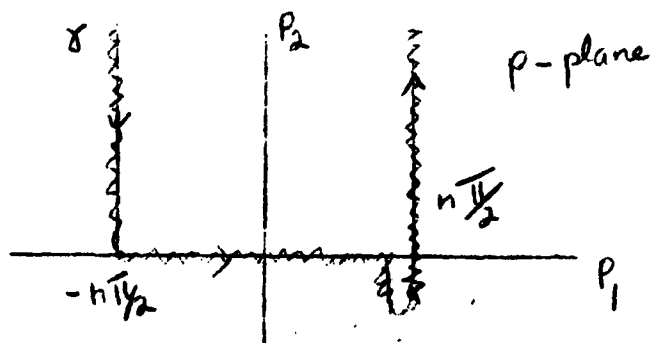


Note that $-\pi/2 \leq \alpha \leq \pi/2$ on C' , a choice of α the reason for which will become apparent below. Since our concern here is with large n , we ought to include n into the change of variable; thus in the integral for $H_n^{(1)}(\rho)$ put $p = \rho \cos \theta + n\theta$. This results in

$$H_n^{(1)}(\rho) = \frac{(-1)^n}{\pi} \int_{\gamma} \frac{e^{ip}}{n - \rho \sin \theta} dp ,$$

where γ is computed from

$$p = p_1 + ip_2 = \cos \alpha \cosh \beta + n\alpha + i(-\sin \alpha \sinh \beta + n\beta):$$



The loop, whose length depends on n , arises from the fact that when $\alpha = \pi/2$ and β goes from 0 to $-\infty$, the quantity $n\beta - \sin \alpha \sinh \beta$ is initially a decreasing function of β .

The integrand above has singularities at $p_k = \rho \cos \theta_0 + n\theta_0 + n(2k\pi)$ if $\sin \theta_0 = \frac{n}{\rho}$ ($k = 0, \pm 1, \dots$). These are arranged on the p -plane in a sequence parallel to the p_1 - axis at intervals of $k(2n\pi)$; we will show in a moment that the p_k are branch points. Thus the contour γ encloses only a single branch point; it is this circumstance which determined the choice of C' , for, as usual, we will wish to apply Cauchy's theorem and close about a vertical branch cut.

Consider $p_0 = \rho \cos \theta_0 + n\theta_0$; we want the behavior of $n - \rho \sin \theta$ in a neighborhood of p_0 , that is, when θ approaches θ_0 . Upon expanding,

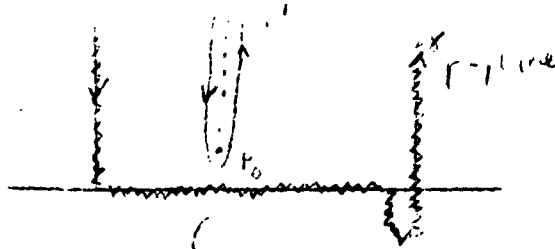
$$\begin{aligned} n - \rho \sin \theta &= -\rho \cos \theta_0 (\theta - \theta_0) + \left(\frac{\rho \sin \theta_0}{2!} \right) (\theta - \theta_0)^2 + \dots \\ p &= \rho \cos \theta_0 + n\theta_0 + (-\rho \sin \theta_0 + n)(\theta - \theta_0) + \dots \\ &\quad + \frac{(-\rho \cos \theta_0)}{2!} (\theta - \theta_0)^2 + \dots \\ &= p_0 - \frac{\rho \cos \theta_0}{2!} (\theta - \theta_0)^2 + \dots \end{aligned}$$

Therefore, near p_0 , $(p - p_0)$ behaves like $-\frac{\rho \cos \theta_0}{2} (\theta - \theta_0)^2$,

and so

$$n - \rho \sin \theta \sim - \left[\frac{2(p-p_0)}{\cos \theta_0} \right]^{1/2} + O(p-p_0) + \dots$$

This indicates that p_0 (and p_k for all k) are branch points. The contour γ contains p_0 ; we take the branch cut vertically, and find that after shifting γ to γ' ,



$$H_n^{(1)}(\rho) = \frac{(-1)^n}{\pi} \int_{\gamma} \frac{e^{i\rho}}{n - \rho \sin \theta} d\rho \sim \frac{2(-1)^n}{\pi} \int_0^{\infty} \frac{e^{i(p_0 + is)}}{\left(\frac{-2is}{\rho \cos \theta_0} \right)^{1/2}} i ds$$

Here $p = p_0 + is$ and the first term in the expansion of $n - \rho \sin \theta$ has been used for the asymptotic estimate. Thus,

$$H_n^{(1)}(\rho) \sim \frac{(-1)^n}{\pi} \sqrt{\frac{2}{i}} \Gamma(1/2) \sqrt{\rho \cos \theta_0} e^{ip_0}.$$

Now, $\frac{n}{\rho} = \sin \theta_0 = \sin \alpha_0 \cosh \beta_0 + i \cos \alpha_0 \cosh \beta_0 \rightarrow \infty$ as n becomes large. Because $\frac{n}{\rho}$ is real for any value of n , we have $\alpha_0 = \pi/2$; referring to the contour C' in the $(\alpha + i\beta) = \theta$ -plane, evidently $\alpha_0 = \pi/2 \Rightarrow \beta_0 < 0$. Thus $\frac{n}{\rho} = \frac{1}{2}(e^{\beta_0} + e^{-\beta_0}) \sim \frac{1}{2}e^{-\beta_0}$ for n large, implying that $\beta_0 \sim -\ln(\frac{2n}{\rho})$.

Also,

$$\rho \cos \theta_0 = \rho(1 - \sin^2 \theta_0)^{1/2} = (\rho^2 - n^2)^{1/2},$$

$$p_0 = \rho \cos \theta_0 + n \theta_0 = (\rho^2 - n^2)^{1/2} + n(\alpha_0 + i\beta_0),$$

which indicates that as n becomes large,

$$\rho \cos \theta_0 \sim \ln$$

$$p_0 \sim \ln - \ln \ln \left(\frac{2n}{\rho} \right) + \frac{n\pi}{2}.$$

Consequently,

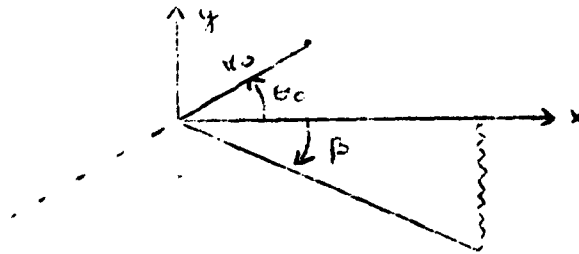
$$H_n^{(1)}(\rho) \sim \frac{(-1)^n}{\pi} \sqrt{\frac{2}{i}} \Gamma(1/2) \sqrt{\ln} e^{i(\ln + n\pi/2 - \ln \ln(\frac{2n}{\rho}))},$$

or,

$$H_n^{(1)}(\rho) \sim \sqrt{\frac{2}{\pi}} (n^{n+1/2} e^{-n}) \left(\frac{\rho}{2} \right)^{-n}.$$

XI. Field produced by a source in the presence of a conducting wedge.

For simplicity, the two dimensional situation will be considered. Assume the wedge to be of infinite length with its edge along the z-axis, and let a sinusoidal line source parallel to the z-axis be given at (r_0, θ_0) .



The electromagnetic field components will be independent of z . From Maxwell's equations (eq.26) T.R. No.2 it is easily seen that two possibilities exist: we may either solve for E_z from which will follow H_x and H_y ; or solve for H_z which also gives E_x and E_y . Both E_z and H_z are seen to satisfy the wave equation

$$(\Delta + k^2) \begin{matrix} E_z \\ H_z \end{matrix} = - \frac{\delta(r-r_0) \delta(\theta - \theta_0)}{r}$$

Supposing the wedge to be a perfect conductor, the boundary conditions are:

$$(i) E_z = 0 \text{ at } \theta = 0 \text{ and } \theta = 2\pi - \beta$$

$$(ii) \frac{\partial H_z}{\partial \theta} = 0 \text{ at } \theta = 0 \text{ and } \theta = 2\pi - \beta$$

Although two distinct problems have been formulated the methods of solution are essentially identical. We shall solve for E_z and in the course of the argument indicate the differences. Of particular interest in the solution will be the shadow region $\pi + \theta_0 \leq \theta \leq 2\pi - \beta$, whose "darkness" varies as a consequence of diffraction at the apex of the wedge. Two other interesting situations are the limiting cases $\beta = \pi$ and $\beta = 0$; the former has already been dealt with in Section IX (reflection off a plane conducting surface), while the latter is the condition for the Sommerfeld half-plane problem (diffraction by a knife edge). The Sommerfeld problem will be discussed below.

As in the preceding section, the wave equation may be written in plane polar form:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k^2 \right) E_z = - \frac{\delta(r-r_0) \delta(\theta-\theta_0)}{r}.$$

The equation is separable; therefore put $-\frac{\partial^2}{\partial \theta^2} = L$. L is an operator acting on the space of functions $u(\theta)$,

$0 \leq \theta \leq 2\pi - \beta$, which satisfy the boundary conditions (1)

above. The eigenfunctions of L are derived from $(\lambda I - L)u = 0$;

they are $\sin \sqrt{\lambda} \theta$, with $\lambda = \left(\frac{n\pi}{2\pi-\beta} \right)^2$. (Were we working with

H_z the only difference here would be that $u(\theta)$ satisfies

the boundary conditions (ii), necessitating the use of cosines instead of sines.) We have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 - \frac{L^2}{r^2} \right) E_z = - \frac{\delta(r-r_0) \delta(\theta-\theta_0)}{r}.$$

The homogeneous equation is Bessel's equation (eq. 21); as in Section I, solutions must be chosen which are regular at the origin and outgoing at infinity. These were labeled $J_{\sqrt{L}}(kr)$ and $H_{\sqrt{L}}^{(1)}(kr)$, respectively, so that

$$\begin{aligned} E_z &= J_{\sqrt{L}}(kr) H_{\sqrt{L}}^{(1)}(kr_0) \alpha & r < r_0 \\ &= J_{\sqrt{L}}(kr_0) H_{\sqrt{L}}^{(1)}(kr) \alpha & r > r_0 \end{aligned}$$

where α is the jump factor. Reference to Section I gives $\alpha = -\frac{\pi i}{2} \delta(\theta-\theta_0)$; however, note that in Section I we found the jump factor by applying the asymptotic forms of the Bessel and Hankel functions to the Wronskian, and that these forms were derived for integral orders of J_r and $H_r^{(1)}$ only. It will be shown in a moment that all the previous representations of J_r and $H_r^{(1)}$ are valid for non-integral r , which will justify $\alpha = -\frac{\pi i}{2} \delta(\theta-\theta_0)$.

To interpret the order \sqrt{L} of the Bessel and Hankel functions, we expand $\delta(\theta-\theta_0)$ in eigenfunctions of L . Letting $\chi = (\frac{\pi}{2\pi-\beta})$ so that $\sqrt{L} = n\chi$, it can be shown that

$$\delta(\theta-\theta_0) = \frac{2\chi}{\pi} \sum_{n=1}^{\infty} \sin n\chi \theta \sin n\chi \theta_0,$$

where $\frac{2\gamma}{\pi}$ is the normalization factor. The operator L appearing in E_z can now, as in Section VIII, Tech. Report No. 2, be replaced by its eigenvalues $n^2\gamma^2$. Thus

$$(15) E_z = i\gamma \sum_{n=1}^{\infty} J_{n\gamma}(kr_<) H_{n\gamma}^{(1)}(kr_>) \sin n\gamma\theta \sin n\gamma\theta_0;$$

$r_> = \max(r, r_0)$ and $r_< = \min(r, r_0)$ has been used. (H_z is given by equation (15) upon the replacement of $\sin n\gamma\theta_{\theta_0}$ by $\cos n\gamma\theta_{\theta_0}$.)

In order to check the convergence of E_z we utilize approximations of $H_{\Delta}^{(1)}$ and J_{Δ} which are valid for large values of Δ . With recourse to the familiar series expansions,

$$J_{\Delta}(\rho) \sim \frac{(\rho/2)^{\Delta}}{\Gamma(\Delta+1)}; \quad H_{\Delta}^{(1)}(\rho) \sim \Gamma(\Delta)(\rho/2)^{-\Delta}, \quad \Delta \gg 1.$$

That is, when Δ is large J_{Δ} and $H_{\Delta}^{(1)}$ behave like the first terms in their expansions. Therefore, for sufficiently large n ,

$$J_{n\gamma}(kr_<) H_{n\gamma}^{(1)}(kr_>) \sim \frac{1}{n\gamma} \left(\frac{r_<}{r_>}\right)^{n\gamma}.$$

Since $\frac{r_<}{r_>} < 1$ wherever $r \neq r_0$, we conclude that the series in equation (7) converges absolutely except on the cylinder $r = r_0$. (Although of no physical significance, it is possible to prove conditional convergence at $r = r_0$, a fact due to the presence of the sine functions.)

The behavior of $J_N(\rho)$ at small values of ρ is again

dominated by the initial term of its series expansion.

Consequently, near the apex of the wedge we may approximate

$$E_z \sim i\gamma \sum_1^{\infty} \frac{(kr/2)^{n\gamma}}{\Gamma(1+n\gamma)} H_n^{(1)}(kr_0) \sin n\gamma \theta \sin n\gamma \theta_0 \sim$$

$$\sim i\gamma \frac{(kr/2)^{\gamma}}{\Gamma(1+\gamma)} H_{\gamma}^{(1)}(kr_0) \sin\gamma \theta \sin\gamma \theta_0, \quad kr \ll 1.$$

This approximation indicates the qualitative behavior of

E_z (and H_z) at the wedge apex. Consider the limiting

cases $\beta = 0$ and $\beta = \pi$, i.e., $\gamma = 1/2$ and $\gamma = 1$:

For $\beta = 0$, $E_z = O(r^{1/2})$ and H_x and H_y , which are proportional to the derivatives of E_z , are $O(r^{-1/2})$. This shows that the Poynting vector $\vec{E} \times \vec{H}$ is bounded as $r \rightarrow 0$ and therefore the energy of the field is finite in the vicinity of the knife edge even though the magnetic field is infinite there.

For $\beta = \pi$, $E_z = O(r)$ and then H_x and H_y will be $O(1)$. This shows that both the electric and magnetic field are finite near $r = 0$. This result was to be expected since the wedge in this case reduces to a plane.

We now turn to an investigation of the field's behavior in general. This will become more feasible if the infinite sum representing E_z can be replaced by a more manageable expression; and as a matter of fact, upon

admitting a certain simplification, we will be able to write E_z as a contour integral.

First, however, the properties of J_γ and $H_\gamma^{(1)}$ which were derived in Section X have to be verified for non-integral values of γ . It was seen that for $\gamma = \text{integer}$,

$$H_\gamma^{(1)}(kr) = \frac{1}{\pi} \int_{C_1} e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} d\varphi$$

$$-\pi \leq \alpha_1 < \alpha_2 \leq \pi, |\alpha_1 - \alpha_2| = \pi.$$

One may verify that this representation of $H_\gamma^{(1)}$ is valid for non-integral γ by showing that

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 - \frac{\gamma^2}{r^2} \right] \int_{C_1} e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} d\varphi = 0.$$

Recalling that \int_{C_1} converges absolutely if $\alpha_2 \neq 0, \pi$, we have

for these values of α_2

$$\begin{aligned} \left[\right]_{C_1} &= -k^2 \int_{C_1} \cos^2 \varphi e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} d\varphi + \dots \\ &\dots + \frac{ik}{r} \int_{C_1} \cos^2 \varphi e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} d\varphi + \end{aligned}$$

$$+ (k^2 - \frac{\gamma^2}{r^2}) \int_{C_1} e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} d\varphi.$$

$$\text{Consider } - \frac{\gamma^2}{r^2} \int_{C_1} :$$

$$- \frac{\gamma^2}{r^2} \int_{C_1} = - \frac{\gamma^2}{r^2} e^{ikr \cos \varphi} \frac{e^{i\gamma(\varphi-\pi/2)}}{i\gamma} \Big|_{C_1} -$$

$$- \frac{\gamma^2}{r^2} \int_{C_1} ikr \sin \varphi e^{ikr \cos \varphi} \frac{e^{i\gamma(\varphi-\pi/2)}}{i\gamma} d\varphi =$$

$$= - \frac{\gamma^2}{r^2} \frac{1}{i\gamma} - \frac{ikr \sin \varphi}{\gamma^2} e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} \Big|_{C_1} + \dots$$

$$\dots + \frac{\gamma^2}{r} \int_{C_1} (-k^2 r^2 \sin^2 \varphi - ikr \cos \varphi) e^{ikr \cos \varphi} \frac{e^{i\gamma(\varphi-\pi/2)}}{\gamma^2}$$

$$\text{when } - \frac{\gamma^2}{r^2} \int_{C_1} \text{ is substituted into } [] \int_{C_1}, \text{ we find}$$

$$[] \int_{C_1} = \text{integrated part} =$$

$$= - \frac{\gamma^2}{r^2} \left(\frac{1}{i\gamma} - \frac{ikr \sin \varphi}{\gamma^2} \right) e^{ikr \cos \varphi} e^{i\gamma(\varphi-\pi/2)} \Big|_{C_1}.$$

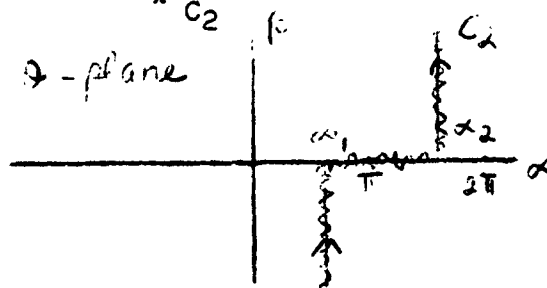
It is not difficult to show that the integrated part is equal to zero. The integral representation of $H_{\gamma}^{(1)}$ is therefore valid for all real γ .

The computation of the asymptotic form of $H^{(1)}$ followed from the integral representation; the argument of pages 12-16 with minor modifications, will also go through for non-integral γ . Referring to equations 15,

$$H_{\gamma}^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4 - \gamma \pi/2)}, \quad kr \gg \gamma.$$

Since $H_{\gamma}^{(1)} = \overline{H_{\gamma}^{(2)}}$, we find as before

$$H_{\gamma}^{(2)}(kr) = \frac{1}{\pi} \int_{C_2} e^{ikr \cos \varphi} e^{-i\gamma(\varphi - \pi/2)} d\varphi.$$



$$0 \leq \alpha_1 < \alpha_2 \leq 2\pi, \quad |\alpha_1 - \alpha_2| = \pi$$

also,

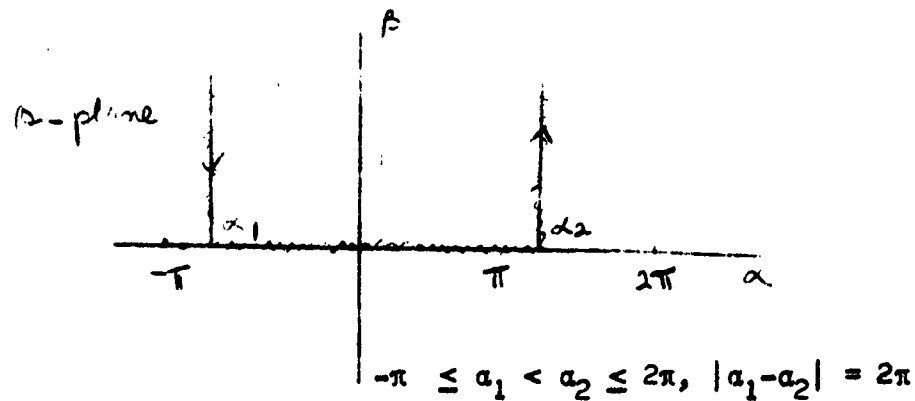
$$H_{\gamma}^{(2)}(kr) \sim \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \pi/4 - \gamma \pi/2)}, \quad kr \gg \gamma$$

The Bessel function J_{γ} may still be expressed as the following linear combination of the Hankel functions:

$$J_{\gamma}(kr) = \frac{1}{2} H_{\gamma}^{(1)}(kr) + \frac{1}{2} H_{\gamma}^{(2)}(kr).$$

But J_{γ} is required to be regular at the origin and this must be verified for the above sum when γ is not an integer. Upon use of the integral representation, it is evident that

$$J_{\gamma}(kr) = \frac{1}{2\pi} \int_{C_3} e^{ikr \cos \varphi} e^{-i\gamma(\varphi - \pi/2)} d\varphi.$$



Thus,

$$J_Y(0) = \frac{1}{2\pi} \int_{C_3} e^{iY(\varphi - \pi/2)} d\varphi$$

which for $Y > 0$ is easily seen to be finite.

Although the integral expressions for J and $H^{(1)}$ are now available for use in equation 15, E_z at best remains an infinite sum of, rather than a single, contour integral. To write E_z in the desired form, and thereby proceed with the examination of its general behavior, the original situation will be slightly simplified. Instead of considering a cylindrical wave incident on the wedge, it will be assumed that the radiation source is a plane wave incoming from the direction θ_0 . In effect, the assumption is tantamount to letting the original line source go to infinity ($r_0 \rightarrow \infty$). Though the source is altered, diffraction still occurs at the wedge apex, the region $\pi + \theta_0 \leq \theta \leq 2\pi - \beta$ continues to lie in the shadow, and a wave is still reflected

off the surface of the wedge. The incident plane wave case therefore presents a good -- and solvable -- analogue to the original wedge problem.

We know the series of equation 15 is absolutely convergent for $r \neq \bar{r}_0$. Since $r_0 \rightarrow \infty$, E_z may be written as

$$E_z = i\gamma \sum_1^N J_{n\gamma}(kr) \sqrt{\frac{2}{\pi k r_0}} e^{i(kr_0 - \pi/4 - n\gamma\pi/2)} \sin n\gamma\theta \sin n\gamma\theta_0 + \epsilon_1$$

from which it will follow that

$$\begin{aligned} E_z &= i\gamma \sum_1^{\infty} J_{n\gamma}(kr) \sqrt{\frac{2}{\pi k r_0}} e^{i(kr_0 - \pi/4 - n\gamma\pi/2)} \sin n\gamma\theta \sin n\gamma\theta_0 = \\ &= \frac{i\gamma}{2\pi} \sqrt{\frac{2}{\pi k r_0}} e^{i(kr_0 - \pi/4)} \int_{C_3} e^{ikr \cos \varphi} \sum_1^{\infty} e^{in\gamma(\varphi - \pi)} \sin n\gamma\theta \sin n\gamma\theta_0 \end{aligned}$$

As $r_0 \rightarrow \infty$, $|E_z| \rightarrow 0$; that is, the amplitude of the field falls off to zero as the source is moved to infinity. Thus, also, the incoming part of E_z , which will be the incident plane wave, loses its amplitude as $r_0 \rightarrow \infty$. In order to avoid this, we multiply E_z by $\sqrt{\frac{\pi k r_0}{2}} e^{-i(kr_0 - \pi/4)}$ -- in effect, E_z is normalized. The z-component of the electric field in the presence of the wedge is therefore

$$F = \sqrt{\frac{\pi k r_0}{2}} e^{-i(kr_0 - \pi/4)} E_z.$$

We get

$$F = \frac{\gamma i}{2\pi} \int_{C_3} e^{ikr \cos \varphi} \sum_1^{\infty} e^{in\gamma(\varphi-\pi)} \sin n\gamma\theta \sin n\gamma\theta_0 d\varphi$$

because we may write

$$\sin n\gamma\theta \sin n\gamma\theta_0 = \frac{1}{4} \left[e^{-in\gamma(\theta-\theta_0)} + e^{-in\gamma(\theta-\theta_0)} - e^{in\gamma(\theta+\theta_0)} - e^{in\gamma(\theta+\theta_0)} \right],$$

F becomes

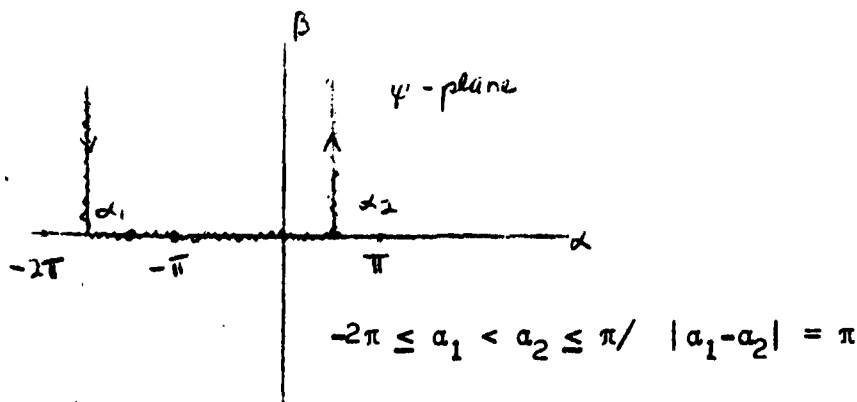
$$\begin{aligned} F &= \frac{\gamma i}{8\pi} \int_{C_3} e^{ikr \cos \varphi} \sum_1^{\infty} \left[e^{(i\gamma(\varphi-\pi+\theta-\theta_0))n} + \dots - e^{(i\gamma(\varphi-\pi-\theta-\theta_0))n} \right] d\varphi, \\ &= \frac{\gamma i}{8\pi} \int_{C_3} e^{ikr \cos \varphi} \left[\frac{e^{i(\varphi-\pi+\theta-\theta_0)}}{1-e^{i(\varphi-\pi+\theta-\theta_0)}} + \dots - \frac{e^{i(\varphi-\pi-\theta-\theta_0)}}{1-e^{i(\varphi-\pi-\theta-\theta_0)}} \right] d\varphi \end{aligned}$$

Thus the field incident on the wedge has been represented by a single contour integral. The sums of the first two terms in the square brackets, and the second two are, respectively

$$\frac{\cos \gamma(\theta-\theta_0) - e^{i\gamma(\varphi-\pi)}}{\cos \gamma(\varphi-\pi) - \cos \gamma(\theta-\theta_0)} \quad \text{and} \quad - \frac{\cos \gamma(\theta+\theta_0) - e^{i\gamma(\varphi-\pi)}}{\cos \gamma(\varphi-\pi) - \cos \gamma(\theta+\theta_0)}.$$

When $\varphi-\pi$ is replaced by φ and the contour C_3 changed accordingly, we have after addition,

$$\begin{aligned} F &= \frac{\gamma}{8\pi} \int_{C_4} e^{-ikr \cos \varphi} \left[\frac{\sin \gamma \varphi}{\cos \varphi - \cos (\theta-\theta_0)} \right] d\varphi \\ &\quad - \frac{\gamma}{8\pi} \int_{C_4} e^{-ikr \cos \varphi} \left[\frac{\sin \gamma \varphi}{\cos \varphi - \cos (\theta+\theta_0)} \right] d\varphi = F_1 + F_2. \end{aligned}$$



Although neither F_1 nor F_2 can be explicitly evaluated, we can nevertheless examine, as usual, their asymptotic behavior and thereby arrive at significant results. Consider F_1 for $\gamma = \frac{1}{2}$ ($\beta = 0 \Rightarrow \gamma = 1/2$; without significant loss of generality, we may reduce the wedge to a knife edge). Then

$$F_1 = \frac{1}{16\pi} \int_{C_4} e^{-ikr \cos \varphi} \left[\frac{\sin \frac{1}{2}\varphi}{\cos \frac{1}{2}\varphi - \cos \frac{1}{2}(\theta - \theta_0)} \right] d\varphi.$$

Let $\gamma = \cos \frac{1}{2}\varphi$, $\gamma_0 = \cos \frac{1}{2}(\theta - \theta_0)$; $d\gamma = -\frac{1}{2}\sin \frac{1}{2}\varphi d\varphi$. Notice that the zeros of $\sin \frac{1}{2}\varphi$ determine branch points in the γ -plane, i.e., $\gamma = \cos \frac{1}{2} 2n\pi = \pm 1$ are branch points. Thus,

$$F_1 = -\frac{2}{16\pi} \int_B \frac{e^{-ikr(2\gamma^2 - 1)} d\gamma}{\gamma - \gamma_0} = -\frac{1}{8\pi} e^{ikr} \int_B \frac{e^{-2ikr\gamma^2}}{\gamma - \gamma_0} d\gamma,$$

where B is to be computed. The contour C_4 may be conveniently taken with $\alpha_1 = -3\pi/2$ and $\alpha_2 = \pi/2$.

- a) When $\varphi = -\frac{3\pi}{2} + i\beta$, $\gamma = \gamma_1 + i\gamma_2 = -\cos \frac{\pi}{4} \cosh \frac{\beta}{2} + i \sin \frac{\pi}{4} \sinh \frac{\beta}{2}$. $\gamma_1/\gamma_2 = -\tan \pi/4 \tanh \beta/2 \rightarrow -1$ as $\beta \rightarrow \infty$; the contour B approaches a slope of -1

in the second quadrant. $\beta = 0 \Rightarrow \zeta = -1/\sqrt{2}$.

b) When $\varphi = \alpha + i\varepsilon$ (the ε is to prevent B from passing through the pole ζ_0):

$$-3\pi/2 \leq \alpha \leq -\pi \Rightarrow -\frac{1}{\sqrt{2}} \leq \zeta_1 \leq 0,$$

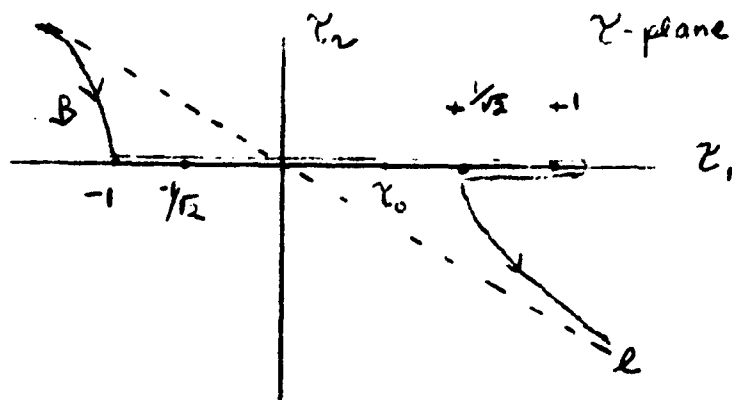
$$-\pi \leq \alpha \leq 0 \Rightarrow 0 \leq \zeta_1 \leq 1,$$

$$0 \leq \alpha \leq \pi/2 \Rightarrow 1 \geq \zeta_1 \geq 1/\sqrt{2},$$

and ζ_2 may be put equal to ε' on this segment of B.

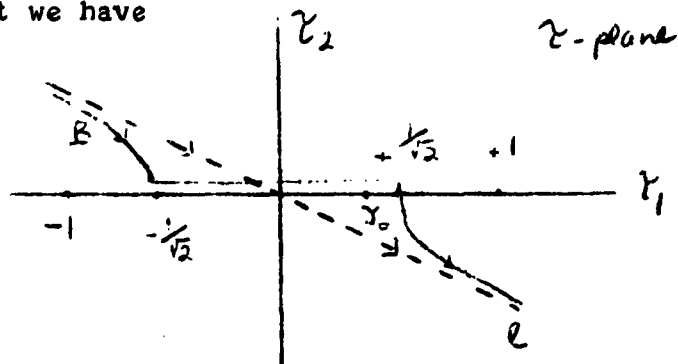
c) When $\varphi = \pi/2 + i\beta$, $\frac{\zeta_2}{\zeta_1} \rightarrow -1$ as $\beta \rightarrow \infty$, so that B also approaches a slope of -1 in the fourth quadrant.

$$\beta = 0 \Rightarrow \zeta = 1/\sqrt{2}.$$



We wish to shift the contour B so that $e^{-2ikr\zeta^2}$ becomes an exponentially decreasing function. The line l in the diagram is suggestive; and in fact on this line

$\mathcal{Z} = Re^{-\pi i/4}$, $-\infty \leq R \leq \infty$, and $\mathcal{Z}^2 = -iR^2$, which is just what's required. The loop about the branch point, however, presents a problem. Cauchy's theorem cannot be applied to the region containing $+1$, and so B , after it is shifted to ℓ , will have to be left with a loop about $+1$. This difficulty can be neatly avoided by noticing that the integrand of F_1 is an odd function of φ , which means that the contributions from $-\pi/2 \leq \varphi \leq 0$ and $0 \leq \varphi \leq \pi/2$ cancel each other. It is just this interval, $-\pi/2 \leq \varphi \leq \pi/2$, which has given rise to the loop in the \mathcal{Z} -plane. Hence, by neglecting it, it follows that we have



The pole \mathcal{Z}_0 will have to be taken into account, that is, when B is shifted to ℓ we will acquire a residue if $\mathcal{Z}_0 > 0$.

The residue at \mathcal{Z}_0 is

$$+ \frac{2\pi i}{8\pi} e^{ikr} e^{-2ikr} \mathcal{Z}_0^2 = + \frac{i}{4} e^{-ikr(2\mathcal{Z}_0^2 - 1)} = + \frac{i}{4} e^{-ikr \cos(\theta - \theta_0)}.$$

The change of sign is due to the orientation of B in the lower

half-plane. Thus,

$$F_1 = -\frac{e^{ikr}}{8\pi} \int_{\mathcal{L}} \frac{e^{-2ikr\zeta^2}}{\zeta - \zeta_0} d\zeta + H(\zeta_0) \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)},$$

where $H(\zeta_0) = 1$ when $\zeta_0 > 0$, zero otherwise. Or, when

$$\zeta^2 = 2kr\gamma^2, \\ F_1 = -\frac{e^{ikr}}{8\pi} \int_{-\infty e^{-\pi i/4}}^{\infty e^{-\pi i/4}} e^{-i\zeta^2} \frac{1}{\zeta - \zeta_0 \sqrt{2kr}} d\zeta + H(\zeta_0) \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)}.$$

In order to estimate the integral in F_1 , we use a technique made familiar in preceding sections. Set

$$\frac{1}{\zeta - \zeta_0 \sqrt{2kr}} = -\frac{1}{\zeta_0 \sqrt{2kr}} + \left[\frac{1}{\zeta_0 \sqrt{2kr}} - \frac{1}{\zeta - \zeta_0 \sqrt{2kr}} \right];$$

then,

$$\begin{aligned} -\frac{e^{ikr}}{8\pi} \int_{-\infty e^{-\pi i/4}}^{\infty e^{-\pi i/4}} e^{-i\zeta^2} \frac{d\zeta}{\zeta - \zeta_0 \sqrt{2kr}} &= \frac{e^{ikr}}{8\pi \zeta_0 \sqrt{2kr}} \int_{-\infty e^{-\pi i/4}}^{\infty e^{-\pi i/4}} e^{-i\zeta^2} d\zeta - \\ -\frac{e^{ikr}}{8\pi} \int_{-\infty e^{-\pi i/4}}^{\infty e^{-\pi i/4}} \frac{e^{-i\zeta^2} \zeta d\zeta}{\zeta_0 \sqrt{2kr} (\zeta - \zeta_0 \sqrt{2kr})} &= \\ &= \frac{e^{i(kr - \pi/4)}}{8 \zeta_0 \sqrt{2\pi kr}} + E_1, \end{aligned}$$

where

$$|E_1| \leq \frac{1}{8\pi} \int_{-\infty}^{\infty} e^{-R^2} \left| \frac{R}{R - e^{\pi i/4} \gamma_0 \sqrt{2kr}} \right| dR \cdot \frac{1}{\gamma_0 \sqrt{2kr}} .$$

The minimum value of the denominator in the integrand of the error term is simply the distance of $e^{\pi i/4} \gamma_0 \sqrt{2kr}$ from the real axis, namely $\gamma_0 \sqrt{kr}$. Therefore

$$|E_1| \leq \frac{1}{8\pi \gamma_0^2 kr} \int_{-\infty}^{\infty} |R| e^{-R^2} dR = O\left(\frac{1}{kr \gamma_0^2}\right) .$$

E_1 is a satisfactory error term so long as γ_0 is not in the neighborhood of zero. We have

$$F_1 = H(\gamma_0) \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)} + \frac{e^{i(kr - \pi/4)}}{8 \gamma_0 \sqrt{2\pi kr}} + E_1 .$$

Similarly for F_2 ; if $\gamma'_0 = \cos \frac{1}{2}(\theta + \theta_0)$,

$$F_2 = -H(\gamma'_0) \frac{1}{4} e^{-ikr \cos(\theta + \theta_0)} - \frac{e^{i(kr - \pi/4)}}{8 \gamma'_0 \sqrt{2\pi kr}} + E_2 .$$

The field produced by the incoming plane wave in the presence of the knife edge is $F_1 + F_2 = F$:

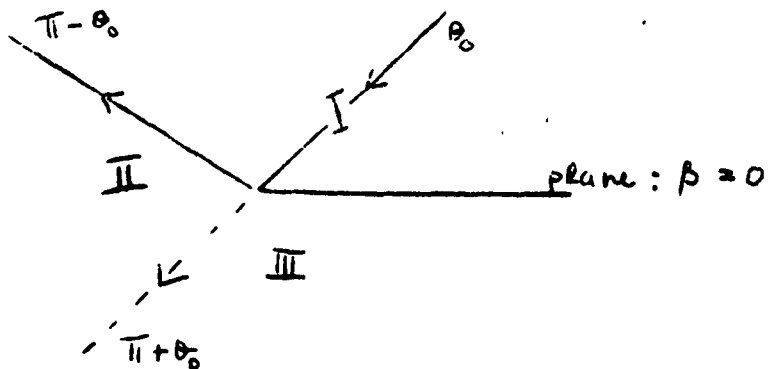
$$(16) F = H(\gamma_0) \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)} - H(\gamma'_0) \frac{1}{4} e^{-ikr \cos(\theta + \theta_0)} + \dots$$

$$\dots + \frac{e^{i(kr - \pi/4)}}{8 \sqrt{2\pi kr}} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma'_0} \right) + E ,$$

where $\gamma_0 = \cos \frac{1}{2}(\theta - \theta_0)$, $\gamma'_0 = \cos \frac{1}{2}(\theta + \theta_0)$,

$$E = 0 \left[\frac{1}{kr} \left(\frac{1}{\gamma_0^2} + \frac{1}{\gamma_0'^2} \right) \right].$$

The physical interpretation of equation (16) is clear. The first two terms of F are plane waves; $\frac{1}{4} e^{-ikr \cos(\theta - \theta_0)}$ is the source wave incoming along the direction θ_0 , and $\frac{1}{4} e^{-ikr \cos(\theta + \theta_0)}$ is incoming along $-\theta_0$, i.e., outgoing along $\pi - \theta_0$, and it represents the reflection of the source by the knife edge. Since $\gamma_0 > 0$ when $0 < \theta < \pi + \theta_0$ and $\gamma_0' > 0$ when $0 < \theta < \pi - \theta_0$, we know in which regions these plane waves exist. The remaining term in equation (16) is a cylindrical wave due to diffraction, and it exists everywhere.



In I, $F = \text{inc.} + \text{outg.} + \text{cyl.} = \text{source} + \text{reflec.} + \text{diff.}$;
 in II, $F = \text{inc.} + \text{cyl.} = \text{source} + \text{diff.}$;
 in III (shadow region), $F = \text{cyl.} = \text{diff.}$

Equation (16) yields no information about the behavior of the field along $\theta = \pi - \theta_0$ and $\theta = \pi + \theta_0$ since there either γ_0 or γ_0' equals zero and the error term breaks down.

In order to determine the field's behavior on the boundary lines, we will have to arrive at an estimate of F which does not depend on the magnitude of γ_0 or γ'_0 .

Consider F_1 ; from before,

$$F_1 = H(\gamma_0) \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)} - \frac{e^{ikr}}{8\pi} \int_{-\infty e^{-\pi i/4}}^{\infty e^{-\pi i/4}} \frac{e^{-t^2}}{t - \gamma_0 \sqrt{2kr}} dt$$

Let $\zeta = R e^{-\pi i/4}$ and $R_0 = e^{\pi i/4} \gamma_0 \sqrt{2kr}$; the integral in F_1 becomes

$$I_1 = -\frac{e^{ikr}}{8\pi} \int_{-\infty}^{\infty} \frac{e^{-R^2}}{R - R_0} dR.$$

Since $R - R_0 = R - \gamma_0 \sqrt{kr} - i \gamma_0 \sqrt{kr}$, we can write

$$\frac{-1}{i(R - R_0)} = \int_0^{\infty} e^{ir(R - R_0)} dr \quad \text{if } \gamma_0 < 0, \quad \text{and}$$

$$\frac{1}{i(R - R_0)} = \int_0^{\infty} e^{-ir(R - R_0)} dr \quad \text{if } \gamma_0 > 0.$$

Therefore, supposing $\gamma_0 < 0$, we have

$$F_1 = I_1 = i \frac{e^{ikr}}{8\pi} \int_{-\infty}^{\infty} e^{-R^2} dR \int_0^{\infty} e^{ir(R - R_0)} dr =$$

$$\begin{aligned}
&= i \frac{e^{ikr}}{8\pi} \int_0^\infty e^{-irR_0} dr \int_{-\infty}^\infty e^{-r^2/4} e^{-(R-i\frac{r}{2})^2} dR = \\
&= i \frac{e^{ikr}}{8\sqrt{\pi}} \int_0^\infty e^{-r^2/4 - irR_0} dr = \\
&= i \frac{e^{ikr}}{8\sqrt{\pi}} e^{-R_0^2} \int_0^\infty e^{-(\frac{r}{2} + iR_0)^2} dr = \\
&= \frac{ie}{4\sqrt{\pi}} e^{ikr(1-2\gamma_0^2)} \int_{iR_0}^{\infty + iR_0} e^{-u^2} du,
\end{aligned}$$

where the last equality was obtained by putting $u = \frac{r}{2} + iR_0$.

In just the same way, when $\gamma_0 > 0$ the other representation of $\frac{1}{R-R_0}$ is used to find

$$F_1 = -\frac{i}{4} e^{ikr(1-2\gamma_0^2)} \left[\frac{1}{\sqrt{\pi}} \int_{-iR_0}^{\infty - iR_0} e^{-u^2} du - 1 \right].$$

The -1 is due to the residue.

For small values of γ_0 , $R_0 \sim 0$; either expression for F_1 gives $F_1 \sim \frac{i}{8} e^{ikr}$ in the neighborhood of $\theta = \pi + \theta_0$. This is simply one-half the incoming plane wave, for if $\theta = \pi + \theta_0$,

$$\frac{1}{2} \frac{i}{4} e^{-ikr \cos(\theta - \theta_0)} = \frac{i}{8} e^{ikr}.$$

A similar result obtains for F_2 : in the neighborhood of $\theta = \pi - \theta_0$, $F_2 \sim \frac{1}{8} e^{ikr}$. Notice that both F_1 and F_2 are outgoing along the regional boundaries.

With reference to equation (16) then, we have:

$$\text{near } \theta = \pi - \theta_0 \ (\chi'_0 \sim 0): F \sim \frac{1}{8} e^{ikr} + \frac{1}{4} e^{-ikr \cos(\theta - \theta_0)} + \frac{e^{i(kr - \pi/4)}}{\chi_0 8\sqrt{2\pi kr}} + O\left(\frac{1}{kr \chi_0^2}\right),$$

$$\text{near } \theta = \pi + \theta_0 \ (\chi_0 \sim 0): F \sim \frac{1}{8} e^{ikr} - \frac{e^{i(kr - \pi/4)}}{\chi_0 8\sqrt{2\pi kr}} + O\left(\frac{1}{kr \chi_0^2}\right),$$

In these expressions, $\chi'_0 \sim -\cos \theta_0$ and $\chi_0 \sim \cos \theta_0$.

Equation (16) gives the field for all other values of θ .